Math 250 4.6 Rolle's Theorem and the Mean Value Theorem

Objectives:

1. **Understand and use Rolle's Theorem**
2. **Understand and use the Mean Value Theorem.**

1) Q: What does it mean if the slopes of the two lines are equal?
A:

2) Q: What does zero slope mean?
A:

3) Q: What is $\frac{f(b)-f(a)}{b-a}$?
A:

4) Q: What is $f'(c)$?
A:

5) Q: If $f'(c)=\frac{f(b)-f(a)}{b-a}$, what does this mean?
A:

6) Q: If $f(b)=f(a)$, what is $\frac{f(b)-f(a)}{b-a}$?
A:

7) Q: If $f'(c)=0$, what does this mean?
A:

Mean Value Theorem: (MVT)
Let *f* be
- continuous on the closed interval [a,b]
- differentiable on the open interval (a,b)

Then:
- there is at least one number c in (a,b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$ = slope of secant line.

Rolle's Theorem: (Special case of the MVT)
Let *f* be
- continuous on the closed interval [a,b]
- differentiable on the open interval (a,b)
- $f(b)=f(a)$

Then:
- there is at least one number c in (a,b) such that $f'(c)=0$.

Math 250 Rolle's Theorem and the Mean Value Theorem

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1. **Understand and use Rolle's Theorem**
2. **Understand and use the Mean Value Theorem.**

1) Q: What does it mean if the slopes of the two lines are equal?
   A: The lines are parallel.

2) Q: What does zero slope mean?
   A: Horizontal line.

3) Q: What is $\frac{f(b)-f(a)}{b-a}$?
   A: Slope of the secant line between $(a, f(a))$ and $(b, f(b))$, m_{sec}

4) Q: What is $f'(c)$?
   A: The slope of the tangent line to *f* at $x = c$, m_{tan}

5) Q: If $f'(c) = \frac{f(b)-f(a)}{b-a}$, what does this mean?
   A: The slope of the tangent line at $x = c$ equals the slope of the secant line between $(a, f(a))$ and $(b, f(b))$, so the secant line and the tangent line at $x = c$ are parallel.

6) Q: If $f(b) = f(a)$, what is $\frac{f(b)-f(a)}{b-a}$?
   A: Zero.

7) Q: If $f'(c) = 0$, what does this mean?
   A: The slope of the tangent line to *f* at $x = c$ is zero, so the tangent line at $x = c$ is horizontal.

Mean Value Theorem: (MVT)
Let *f* be

- continuous on the closed interval [a,b]
- differentiable on the open interval (a,b)

Then:

- there is at least one number c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ = slope of secant line.

Rolle's Theorem: (Special case of the MVT)
Let *f* be

- continuous on the closed interval [a,b]
- differentiable on the open interval (a,b)
- $f(b) = f(a)$

Then:

- there is at least one number c in (a,b) such that $f'(c) = 0$.

Math 250 Rolle's Theorem and the Mean Value Theorem Briggs 4.6

Objectives:

1. **Understand and use Rolle's Theorem**
2. **Understand and use the Mean Value Theorem.** (MVT)

(1) Q: What does it mean if the slopes of the two lines are equal?
A: The lines are parallel.
} required for MVT

(2) Q: What does zero slope mean?
A: The line is horizontal.
} required for Rolle's

(3) Q: What is $\frac{f(b)-f(a)}{b-a}$?
A: Slope of secant line through $(a, f(a))$ and $(b, f(b))$
} required for MVT (implied in Rolle's)

(4) Q: What is $f'(c)$?
A: Slope of tangent line at $x=c$.
} required for both MVT and Rolle's

(5) Q: If $f'(c)=\frac{f(b)-f(a)}{b-a}$, what does this mean?
ESSENTIAL POINT OF MVT

* A: Slope of tangent line at $x=c$ equals slope of secant line through $(a, f(a))$ and $(b, f(b))$, so the tangent at $x=c$ is parallel to the secant through $(a, f(a))$ and $(b, f(b))$.

The difference between MVT and Rolle's

Q: If $f(b)=f(a)$, what is $\frac{f(b)-f(a)}{b-a}$?

(6) A: $\frac{0}{b-a}=0$ so slope of secant through $(a, f(a))$ and $(b, f(b))$ is zero, making this secant horizontal.

Applies only to Rolle's

(7) Q: If $f'(c)=0$, what does this mean?
A: The slope of tangent at $x=c$ is 0, so this tangent is horizontal.

Mean Value Theorem: (MVT)

Let f be

- continuous on the closed interval [a,b]
- differentiable on the open interval (a,b)

} If these are true, then the MVT can be applied. Must memorize and check each.

Then:

- there is at least one number c in (a,b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$ = slope of secant line. } This is a calculation

Rolle's Theorem: (Special case of the MVT)

Let f be

- continuous on the closed interval [a,b]
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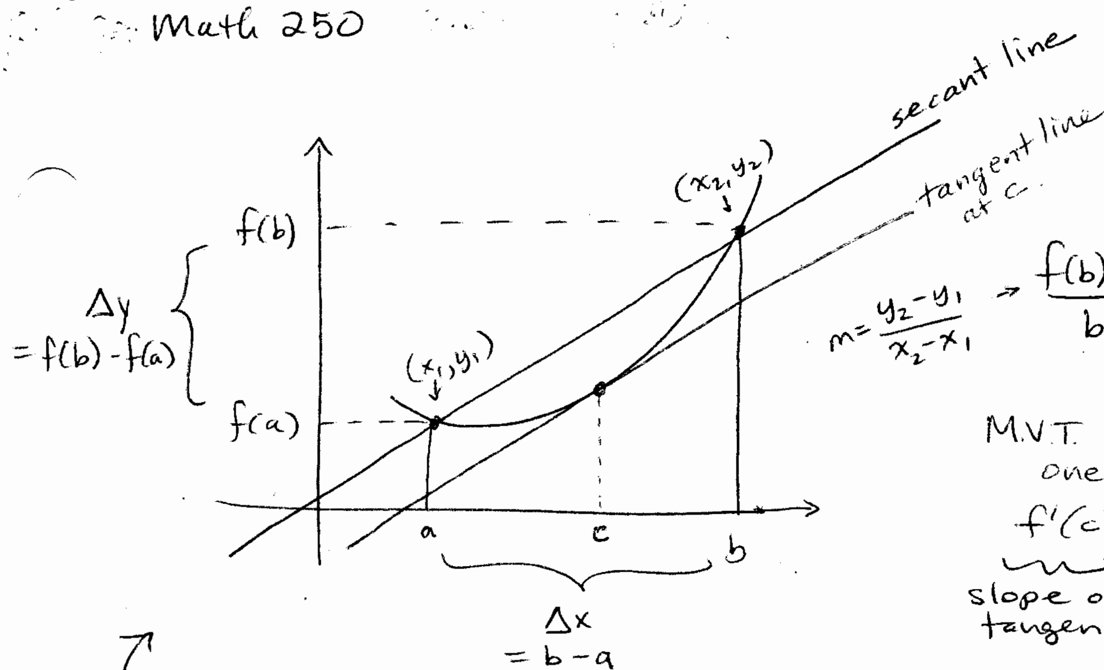
- $f(b)=f(a)$

} If these are true, then Rolle's can be applied. Must memorize and check each.

Then:

- there is at least one number c in (a,b) such that $f'(c)=0$. } This is a calculation.



$m = \dfrac{y_2 - y_1}{x_2 - x_1} \rightarrow \dfrac{f(b) - f(a)}{b - a} = \dfrac{\Delta y}{\Delta x} =$ slope of secant line

M.V.T. says there is at least one point c so that

$$\underbrace{f'(c)}_{\text{slope of tangent}} = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\text{slope of secant}}$$

Mean Value Theorem picture because $f(a) \neq f(b)$.

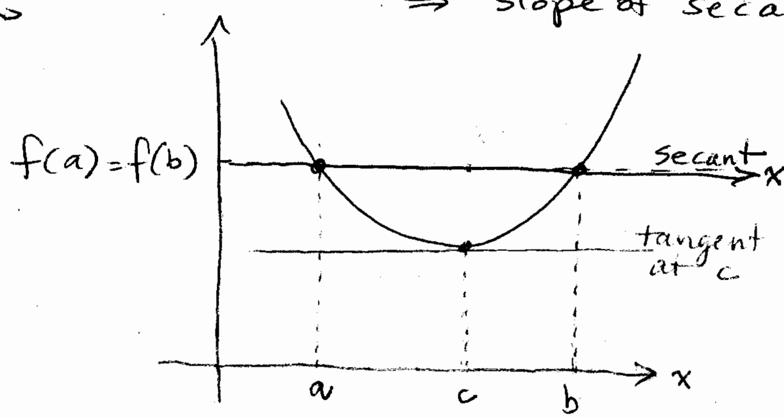
$\Rightarrow$ slope of secant line is NOT 0.

Both theorems:

$\boxed{\text{At } c\text{, the tangent line is parallel to the secant line. (same slope)}}$

Rolle's Theorem picture because $f(a) = f(b)$

$\Rightarrow$ slope of secant line is 0.



$f(a) = f(b)$ means

$\Delta y = f(b) - f(a) = 0$.

$\Rightarrow \dfrac{f(b) - f(a)}{b - a} = \dfrac{0}{b - a} = 0$

$\Rightarrow$ slope of secant is 0

$\Rightarrow$ horizontal secant.

Rolle's Theorem says there is at least one point c so that $f'(c) = \dfrac{f(b) - f(a)}{b - a} = 0$.

Buried in the statements $f(a) = f(b)$

and $f'(c) = 0$

are the concepts that the secant is parallel to the tangent and both are horizontal.

M250

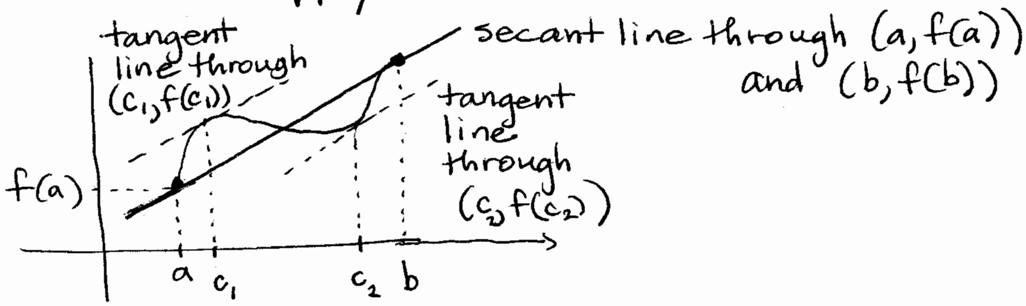
Mean Value Theorem (MVT) * Memorize

IF:
1) f is continuous on $[a,b]$
   2) Why is f continuous?
      ex: f is a polynomial, polys continuous $(-\infty,\infty)$
      ex: f is a trig with no asymptotes in $[a,b]$.

3) f is differentiable on (a,b)
   4) Why is f differentiable?
      ex: f is a poly, differentiable $(-\infty,\infty)$
      ex: f is a trig w/ no asymptotes in (a,b)

If we can do 1)–4) above
Then we can apply the MVT.

THEN:



Then there exists at least one x-value $x=c$ so that c is in (a,b):

1) Find $m_{sec} = \dfrac{f(b)-f(a)}{b-a}$

2) Find $f'(x) = m_{TAN}$

3) Set equal and solve

$$\underbrace{f'(c)}_{\text{slope of tangent to } f \text{ at } x=c} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of secant through } (a,f(a)) \text{ and } (b,f(b))}$$

Recall: if slope l_1 = slope l_2 then l_1 is parallel to l_2

Math 250

When doing a Rolle's or MVT problem:

First, memorize the if and then statements carefully, with attention to open (a,b) and closed $[a,b]$.

Second, in your work

- List the if statement
- Give the reason the if statement is true.
  - ex: polynomials are continuous everywhere
  - ex: trigs (or rationals) are differentiable everywhere except at asymptotes. No asymptote is in the interval.
  - ex: calculate $f(a)$ and $f(b)$, show $f(a)=f(b)$.
- Repeat for all if statements.

Third, in your work
state whether all if statements were true (and the theorem can be applied)
OR that one or more statements were false (and the theorem cannot be applied).

Last, write the then statement, and if requested, do any calculations.

Rolle's	MVT
• Solve $f'(c)=0$ $\{$or $f'(x)=0\}$	• Find $m_{sec} = \dfrac{f(b)-f(a)}{b-a}$
	• Find $m_{tan} = f'(x)$ $\{$or $f'(c)\}$
	• Set results equal and solve for x (or c).

M250 3.2 "Determine whether Rolle's Theorem can be applied."

Verify if the hypotheses of Rolle's Theorem are satisfied.
If yes, find all values of c

(1) $f(x) = x^2 - 8x + 5 \quad [2,6]$

continuous: f is a polynomial, continuous everywhere ✓

differentiable: f is a polynomial, differentiable everywhere ✓

$f(a) = f(b)$: $\quad f(2) = 2^2 - 8(2) + 5 = -7$

$\qquad f(6) = 6^2 - 8(6) + 5 = -7$

$\qquad f(2) = f(6) = 7$ ✓

Yes, the hypotheses are satisfied.

$f'(x) = 2x - 8$ = slope of tangent

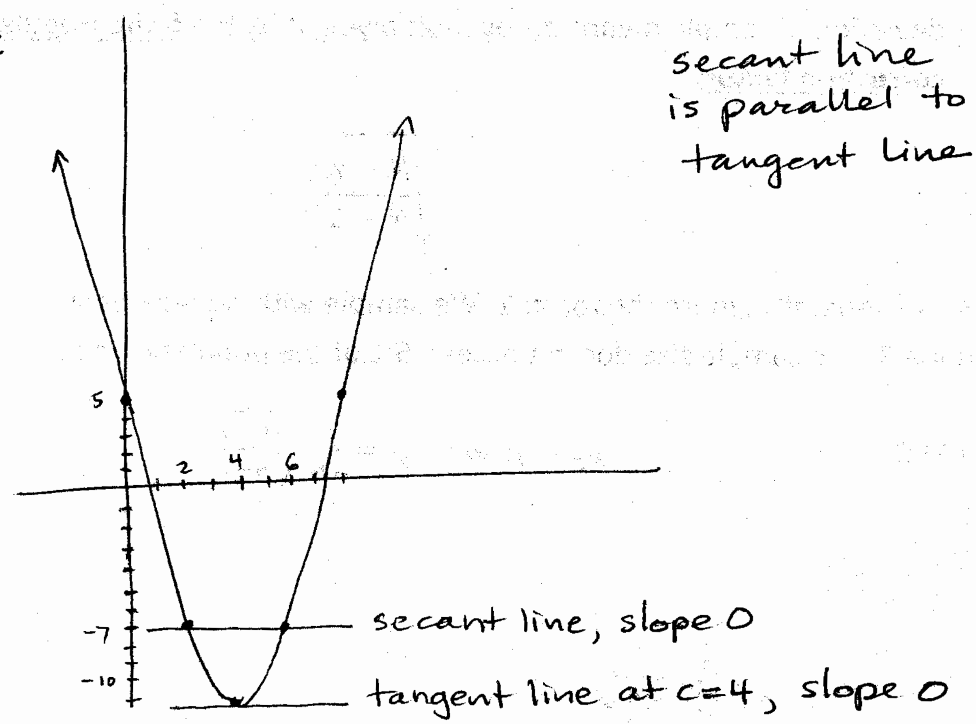
$\dfrac{f(b) - f(a)}{b - a} = \dfrac{-7-(-7)}{6-2} = \dfrac{0}{4} = 0$ = slope of secant

$2c - 8 = 0 \qquad m_{TAN} = m_{sec}$

$2c = 8$

$\boxed{c = 4}$

Graphically:



"Determine whether the MVT can be applied."

Verify that the hypotheses of the Mean Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

(2) $f(x) = x - \frac{1}{x}$ $[3,4]$.

f continuous everywhere except $x=0$
b/c rational functions are continuous whenever their denominators $\neq 0$.

$x=0$ is not in $[3,4]$ so f is cont on $[3,4]$. ✓

f diff on $(3,4)$?

$f(x) = x - x^{-1}$

$f'(x) = 1 + x^{-2} = 1 + \frac{1}{x^2} = \frac{x^2+1}{x^2}$ diff everywhere except $x=0$

b/c rational functions are differentiable whenever their denominators $\neq 0$.

$x=0$ is not in $(3,4)$ so f is diff on $(3,4)$ ✓

So $\boxed{\text{Yes}}$ — the MVT can be applied.

step 1: find $f'(x) = 1 + x^{-2}$

step 2: find $m_{sec} = \frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(3)}{4-3} = \frac{\frac{15}{4} - \frac{8}{3}}{1} = \frac{13}{12}$

step 3: Set $f'(x) = m_{sec}$ and solve

$$1 + x^{-2} = \frac{13}{12}$$

$$\frac{1}{x^2} = \frac{1}{12}$$

$$x^2 = 12$$

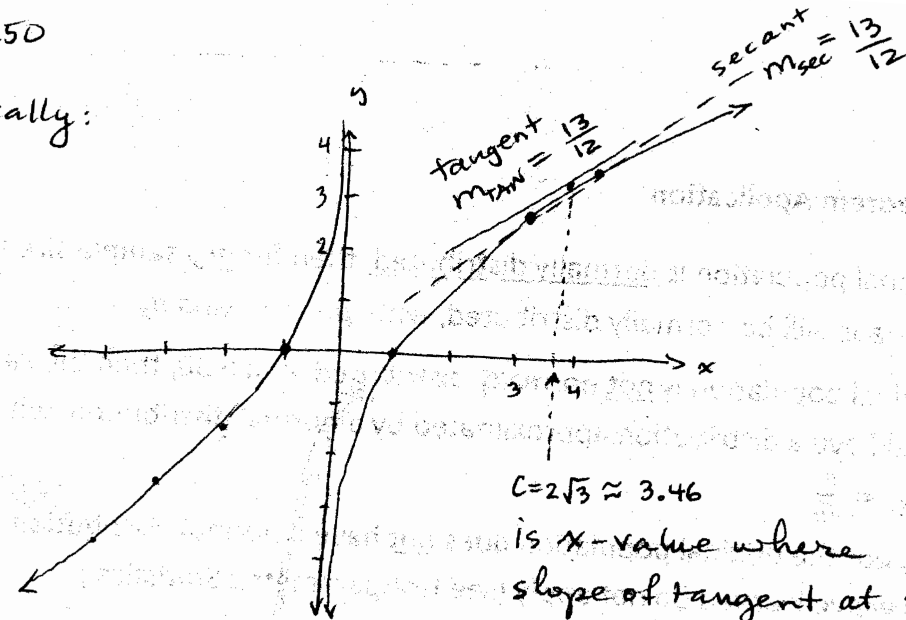
$$x = \pm\sqrt{12}$$

$$x = \pm 2\sqrt{3} \approx \pm 3.46$$

$\boxed{x = 2\sqrt{3}}$ or $\boxed{c = 2\sqrt{3}}$

Math 250

Graphically:



$c = 2\sqrt{3} \approx 3.46$
is x-value where
slope of tangent at $2\sqrt{3}$
equals
slope of secant through
$(3, \frac{8}{3})$ and $(4, \frac{15}{4})$.

Two lines whose slopes
are equal are parallel lines.

Determine whether Rolle's Theorem can be applied on the given interval. If yes, find all values of c satisfying Rolle's Theorem. If no, explain.

(3) $f(x) = \dfrac{x^2 - 2x - 3}{x+2}$ on $[-1, 3]$

f is cont except at $x = -2$ which is not in the interval.
f is diff on interval
$f(-1) = 0$
$f(3) = 0$.
Rolle's Theorem can be applied.

$$f'(x) = \frac{(x+2)(2x-2) - (x^2-2x-3)(1)}{(x+2)^2}$$

$$= \frac{2x^2+2x-4 - x^2+2x+3}{(x+2)^2}$$

$$= \frac{x^2+4x-1}{(x+2)^2}$$

$f'(x) = 0$ when $x^2 + 4x - 1 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4(1)(-1)}}{2}$$

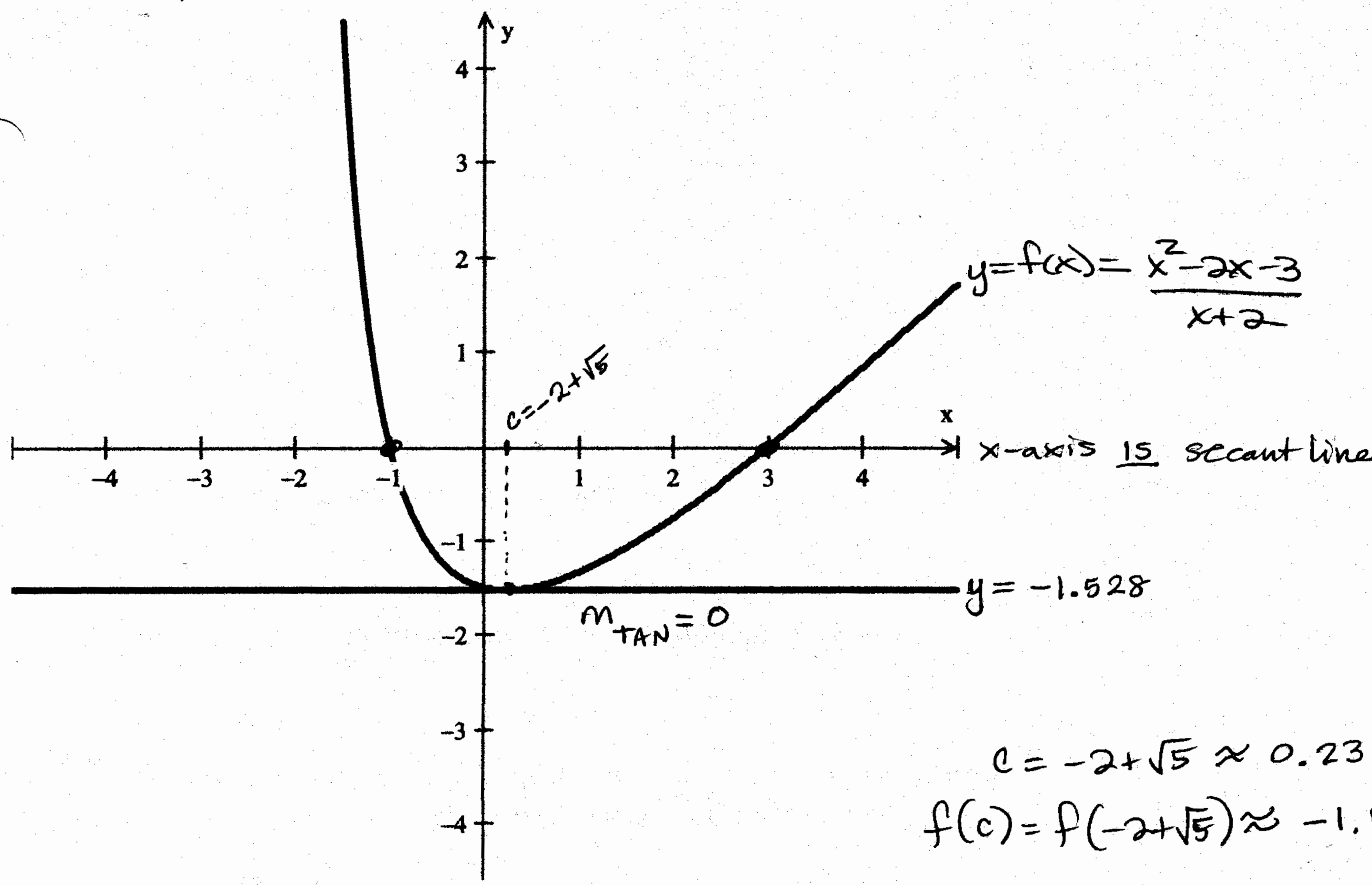
$$= \frac{-4 \pm \sqrt{20}}{2}$$

$$= \frac{-4}{2} \pm \frac{2\sqrt{5}}{2}$$

$$= -2 \pm \sqrt{5} \approx .2361, \quad -4.2361 \text{ not in interval}$$

$\boxed{c = -2 + \sqrt{5}}$ gives $f'(c) = 0$.

③, continued



$c=-2+\sqrt{5}\approx 0.236$

$f(c)=f(-2+\sqrt{5})\approx -1.528$

④ $f(x)=(x+2)^{2/3}$ on $[-3,0]$.

f is cont on $[-3,0]$.

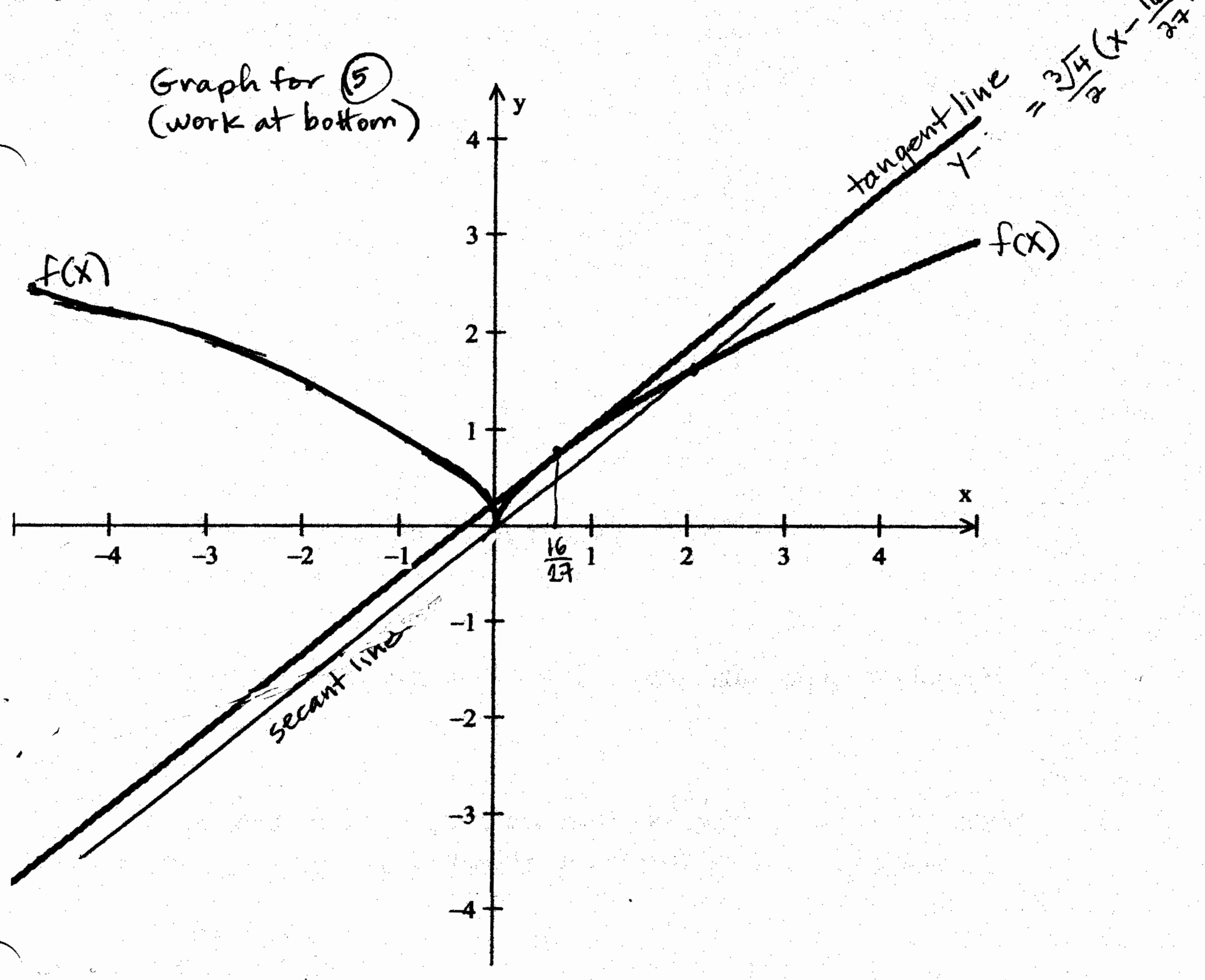
f is not diff at $x=-2$. } Rolle's Theorem cannot be applied.

$f(-3)=1$

$f(0)=\sqrt[3]{4}$ } Rolle's Theorem cannot be applied.

Either of the two reasons prevent use of Rolle's Thm.

Graph for (5)
(work at bottom)



(5) $f(x) = x^{2/3}$ on $[0,2]$

f cont on $[0,2]$ b/c cont everywhere ($\sqrt[3]{\ }$)

f diff on $(0,2)$ b/c diff everywhere ($\sqrt[3]{\ }$) except $x=0$, not in $(0,2)$.

M.V.T. can be applied.

$f(0) = 0$

$f(2) = \sqrt[3]{4}$

$$\frac{f(2)-f(0)}{2-0} = \frac{\sqrt[3]{4}-0}{2-0} = \frac{\sqrt[3]{4}}{2} = \text{slope of secant line.}$$

Solve $\underset{m_{tan}}{f'(x)} = \frac{\sqrt[3]{4}}{2}$ ← m_{sec}

$$f'(x) = \tfrac{2}{3}x^{-1/3} = \overbrace{\frac{2}{3\sqrt[3]{x}} = \frac{\sqrt[3]{4}}{2}}^{\text{solve}}$$

$$\left(\frac{4}{3\sqrt[3]{4}}\right)^3 = \left(\sqrt[3]{x}\right)^3$$

$$\frac{16}{27} = \frac{64}{27 \cdot 4} = x$$

$\boxed{c = \frac{16}{27}} = .\overline{592}$

⑥ $f(x) = x^{2/3}$ on $[-1, 2]$ $\qquad$ $f'(x)$ = same as ⑤

vertical tangent at $x=0$.

f is cont on $[-1, 2]$

f is not diff at $x=0$, in the interval $(-1, 2)$

$\boxed{\text{MVT cannot be applied b/c of non-diff at } x=0}$

⑦ Determine whether the M.V.T. can be applied on the given interval. If yes, find all values of c. If no, explain.

$f(x) = x^4 - 8x$ on $[0, 2]$

f diff and cont
$(0,2)$ $\quad$ $[0,2]$.

$f(a) = f(0) = 0$

$f(b) = f(2) = 0$

$\boxed{\text{yes — the MVT is Rollé's Theorem in this case.}}$

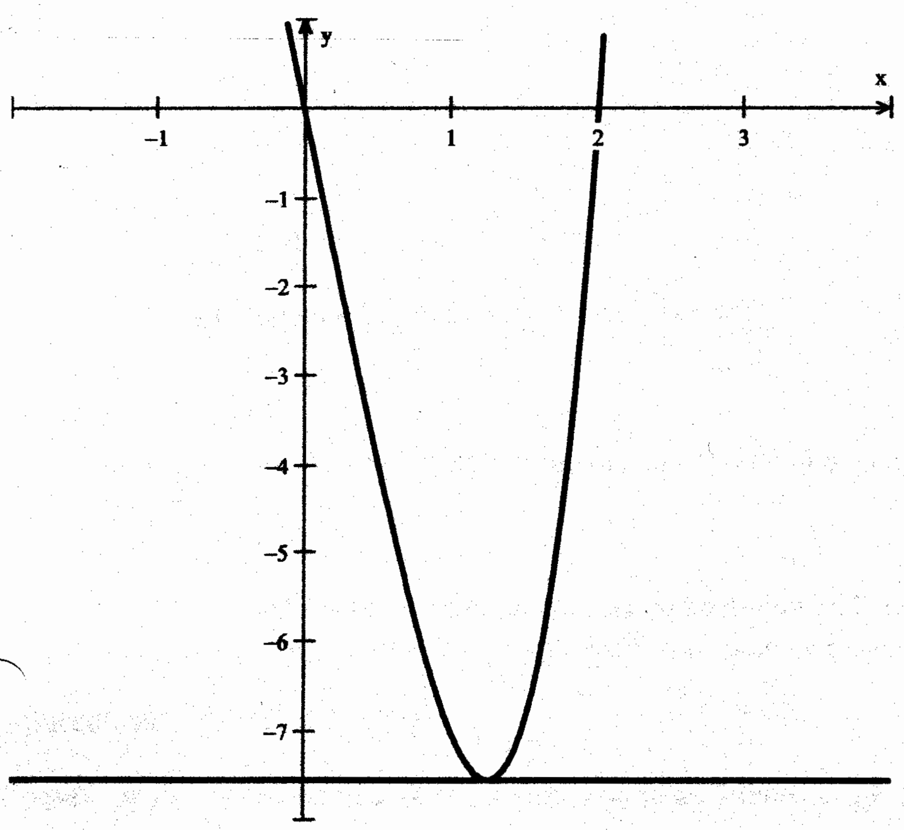
$f'(x) = 4x^3 - 8$

$4x^3 - 8 = 0$

$x^3 - 2 = 0$

$x = \sqrt[3]{2}$

$\boxed{c = \sqrt[3]{2}}$



Math 250 Briggs 4.6 MVT + Rolle's

Thm 4.10 Zero derivative Implies Constant Function
If f is differentiable on an interval I
then f is a constant on interval I.

Proof:
If $f'(x)=0$ on $[a,b]=I$ then MVT says there is a point c in $a,b)$ so that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

$$\frac{f(b)-f(a)}{b-a}=0$$

mult both sides by $(b-a)$

$$\cancel{(b-a)}\frac{f(b)-f(a)}{\cancel{(b-a)}}=0\cdot(b-a)$$

$$f(b)-f(a)=0$$

$$f(b)=f(a).$$

We have the same function value, so $f(x)=$ that value for any values of x in (a,b).

Why do we care?

We know that from the power rule

if $f(x)=$ constant then $f'(x)=0$.

But now we can reverse the logic

if $f'(x)=0$ then $f(x)=$ constant.

Math250 Briggs 4.6 MVT & Rolle's

Thm 4.11 Functions with Equal Derivatives Differ By a Constant

If two functions f and g have $f'(x) = g'(x)$ for all x on

then $f(x) - g(x) =$ constant C for all x on I.

"f and g differ by a constant".

Proof: $f'(x) = g'(x)$ for x in I

so $f'(x) - g'(x) = 0$ if we subtract $g'(x)$ from both sides

This means $\frac{d}{dx}[f] - \frac{d}{dx}[g] = 0$ change of notation

and $\frac{d}{dx}[f-g] = 0$ sum rule for derivatives (backward direction).

so $f - g =$ constant C by Thm 4.10!

The difference between f(x) and g(x),

$f(x) - g(x)$

is a constant for all x in I.

M250

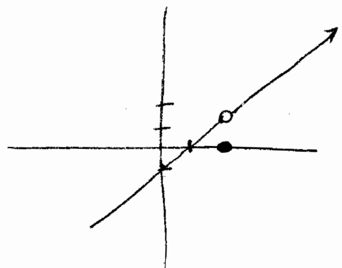
Why is it required that the function f be continuous on the closed interval? Wouldn't an open interval do?

No.

Rolle's fails for cont on open interval

$$f(x) = \begin{cases} x-1 & x \neq 2 \\ 0 & x = 2 \end{cases}$$

on $[1,2]$.



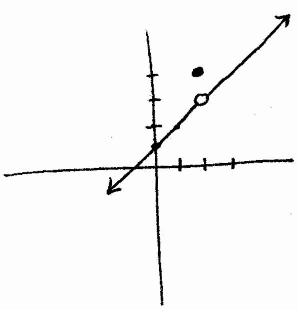
$f(1) = f(2) = 0$

$$f'(x) = \begin{cases} 1 & x \neq 2 \\ \text{undef} & x = 2 \end{cases}$$

$f'(x) = 0$ nowhere

MVT fails.

$$f(x) = \begin{cases} x+1 & x \neq 2 \\ 4 & x = 2 \end{cases}$$



Cont $[0,2)$
diff $(0,2)$

$f(2) = 4$
$f(0) = 1$

$$m_{sec} = \frac{4-1}{2-0} = \frac{3}{2}$$

$$f'(x) = \begin{cases} 1 & x \neq 2 \\ \text{undef} & x = 2 \end{cases}$$

$3/2 \neq 1 \quad 3/2 \neq 0$

no value of c exists.

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(8) Verify if the hypotheses of the Mean Value Theorem are satisfied. If yes, find all values of c guaranteed by the theorem.

$f(x) = \cos(x)$ on $\left[-\frac{\pi}{6}, \frac{\pi}{4}\right]$

Verify "if" statements:

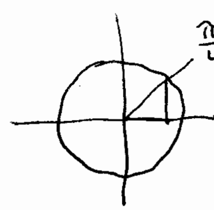
1) Is $f(x)=\cos(x)$ continuous on $\left[-\frac{\pi}{6}, \frac{\pi}{4}\right]$?
   yes, because $\cos x$ is a trig, continuous everywhere because it has no asymptotes.

2) Is $f(x)=\cos(x)$ differentiable on $\left(-\frac{\pi}{6}, \frac{\pi}{4}\right)$?
   yes, because $\cos x$ is a trig, differentiable everywhere (it has no asymptotes).

$\boxed{\text{Yes, MVT can be applied.}}$

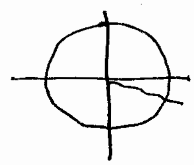
Calculation permitted by "then" statements:

$$m_{sec} = \frac{f(b)-f(a)}{b-a} = \frac{\cos\left(\frac{\pi}{4}\right) - \cos\left(-\frac{\pi}{6}\right)}{\frac{\pi}{4} - \left(-\frac{\pi}{6}\right)}$$



$\frac{\pi}{4} \Rightarrow \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$= \frac{\left(\frac{\sqrt{2}}{2} - \frac{+\sqrt{3}}{2}\right)}{\left(\frac{\pi}{4} + \frac{\pi}{6}\right)}$$



$-\frac{\pi}{6} \quad \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

$$= \frac{\left(\frac{\sqrt{2}-\sqrt{3}}{2}\right)}{\left(\frac{5\pi}{12}\right)} = \frac{6(\sqrt{2}-\sqrt{3})}{5\pi}$$

$m_{TAN} = f'(x) = -\sin(x)$ or $f'(c) = -\sin(c)$

Solve $\quad -\sin(x) = \dfrac{6(\sqrt{2}-\sqrt{3})}{5\pi}$

← This value is not an easy one from a known angle. We can either
- write an exact answer in terms of $\sin^{-1}$

OR
- write an approximate answer.

$$\underset{\text{angle } x}{\sin(x)} = \underset{\text{sine ratio}}{\frac{6(\sqrt{3}-\sqrt{2})}{5\pi}}$$

Exact: $\quad c_1 = x = \sin^{-1}\left(\dfrac{6(\sqrt{3}-\sqrt{2})}{5\pi}\right)$

Is this within $\left(-\frac{\pi}{6}, \frac{\pi}{4}\right)$?

Math 250

(8 cont) To check if $c_1 = \sin^{-1}\left(\frac{6(\sqrt{3}-\sqrt{2})}{5\pi}\right)$ is in $\left(-\frac{\pi}{6}, \frac{\pi}{4}\right)$,

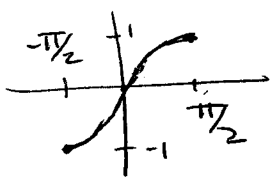
approximate each value.

$$-\frac{\pi}{6} \approx -.5235987756$$

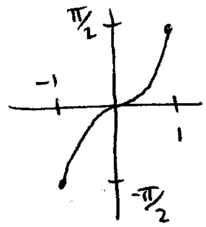
$$\frac{\pi}{4} \approx .7853981634$$

check $\boxed{\text{MODE}}$ is radians! $\longrightarrow$ $\boxed{\sin^{-1}\left(\frac{6(\sqrt{3}-\sqrt{2})}{5\pi}\right) \approx .1217051145 = c_1}$ yes!

But wait! Is there another angle besides $\sin^{-1}\left(\frac{6(\sqrt{3}-\sqrt{2})}{6\pi}\right)$?

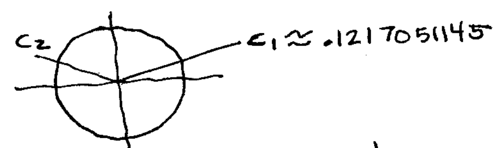
Recall $\sin(x)$ 

is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to get

$\sin^{-1}(x)$ 

and the GC gives only angles within that restriction.

The sine ratio we have is $\frac{6(\sqrt{3}-\sqrt{2})}{5\pi} \approx .1214048848$, positive.

$\sin(x)$ is positive in QI and QII 

$c_2 = \pi - c_1$ has the same sine value!

$\sin(c_2) = \sin(\pi - c_1) = \sin(c_1). = \frac{6(\sqrt{3}-\sqrt{2})}{5\pi}.$

Do we care? No, because c_2 is not in interval $\left(-\frac{\pi}{6}, \frac{\pi}{4}\right)$!
It's in QII, from $\frac{\pi}{2}$ to π.

Out of curiosity, how big is c_1 in degrees?

$c_1 = \sin^{-1}\left(\frac{6(\sqrt{3}-\sqrt{2})}{5\pi}\right)$ radians $\approx .1217051145$ radians

Convert to degrees $\sin^{-1}\left(\frac{6(\sqrt{3}-\sqrt{2})}{5\pi}\right) \cdot \frac{180}{\pi} \approx 6.9731^\circ \approx 7^\circ$

$\frac{\text{radians}}{\pi} = \frac{\text{degrees}}{180}$